Chapter 1

The Chinese Rings

In this chapter, we will discuss the mathematical theory of the CR which goes back to the booklet by Gros of 1872 [119]. This mathematical model may serve as a prototype for the approach to analyze other puzzles in later chapters. In Section 1.1 we develop the theory based on binary coding leading to a remarkable sequence to be discussed in Section 1.2. Some applications will be presented in Section 1.3.

1.1 Theory of the Chinese Rings

Recall the initial appearance of the puzzle with all rings on the bar (cf. Figure 1.1).

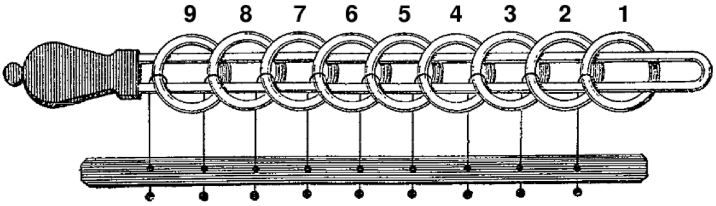

Figure 1.1: Chinese rings

In the introduction we assumed that all configurations of the system of rings can be reached from this initial state using just two kinds of individual ring moves, which we will now specify as:

- the rightmost ring can always be moved (*move type* 0),
- the ring after the first ring on the bar (from the right) can be moved (*move type* 1);

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we have chosen the handle to be on the left for mathematical reasons, the rightmost ring having the least “place value”. From a practical point of view, a move of type 0 consists of two steps, or movements in the sense of Wallis (cf. Section 0.6.1): to move the rightmost ring off the bar, one has to pull the bar back and then the ring through the loop; for every other ring, the bar has to be pulled back through its right-hand neighbor and itself (two movements), then the moving ring goes through the loop and finally the bar forward through the neigboring ring, all in all four movements. Since all the movements of the bar are “forced” by the material, we will only count moves of rings, i.e. movements through the loop.

The original task was to get all rings off the bar, and we have to show that our assumption was correct. The questions arising are

- Is there a solution? (If the answer is “yes”, then there is also a shortest solution with respect to the number of moves needed.)
- Is there only one (shortest) solution?
- Is there an *efficient* solution, i.e. an algorithm realizing the shortest solution?

The same types of questions will be asked for other puzzles in later chapters.

For every ring there are two conditions, namely to be off or on the bar, represented by 0 and 1, respectively. We define $B := [2]_0 = \{0, 1\}$.

Then every state of the CR with $n \in \mathbb{N}_0$ rings can be represented by an $s = s_n \dots s_1 \in B^n$, where $s_r = 0$ (1) means that ring $r \in [n]$ (numbered from right to left) is off (on) the bar. (We include the case $n = 0$ for technical reasons; $s \in B^0$ is the empty string then by convention.) According to the rules, a bit s_{k+1} can be switched, if either $k = 0$ (move type 0) or $s_k = 1$ and $\forall\, l \in [k-1] : \ s_l = 0$ (move type 1). That is to say, with $b \in B$,

- any state $x \dots yb$ can be transformed into $x \dots y(1-b)$ (move type 0),
- any state $x \dots yb10 \dots 0$ into $x \dots y(1-b)10 \dots 0$ (move type 1).

For $n \in \mathbb{N}_0$ the task translates to finding a (shortest) path from 1^n to 0^n in the graph R^n whose vertex set is B^n and whose edges are formed by pairs of states differing by the legal switch of one bit. (Here and in the sequel the string $b \dots b$ of length $k \in \mathbb{N}_0 \cup \{\infty\}$ will be written as b^k; similarly, for strings s and t, st will denote the concatenated string.) Since this graph is undirected, we may as well start in $\alpha^{(n)} := 0^n$, a vertex of degree 1, if $n \in \mathbb{N}$. Its only neighbor is $0^{n-1}1$. If $n > 1$, the next move is either to go back to 0^n, which in view of a *shortest* path to 1^n is certainly not a good idea, or to the only other neighbor $0^{n-2}11$. Continuing this way we will never return to a vertex already visited, because their degrees (at most 2) have been used up already. Therefore, we finally arrive on this **path graph** at the only other vertex of degree 1, namely $\omega^{(n)} := 10^{n-1}$. (In addition, we define $\omega^{(0)}$ as the empty string for definiteness as before.) This does, however, not guarantee that we passed the goal state 1^n on our way! A graph with vertices of degree 2 except for exactly two **pendant vertices** consists of a path and a certain number of **cycles** (cf. Exercise 1.1). So what we have to show is that R^n is connected.

Theorem 1.1. *The graph R^n is connected for every $n \in \mathbb{N}_0$. More precisely, R^n is the path on 2^n vertices from $\alpha^{(n)}$ to $\omega^{(n)}$.*

Proof. The proof is by induction on n. The case $n = 0$ is trivial. For $n \in \mathbb{N}_0$ we know by induction assumption that R^n is the path from $\alpha^{(n)}$ to $\omega^{(n)}$. Attaching 0 at the left of each of its vertices we get a path from $\alpha^{(1+n)}$ to $0\omega^{(n)}$ in R^{1+n} which passes through all states that start with 0. Similarly, attaching 1 to the vertices of the same path but taken in reverse order, gives a path on 2^n vertices in R^{1+n} between $1\omega^{(n)}$ and $\omega^{(1+n)}$. Since these two paths are linked in R^{1+n} by precisely one edge, namely the edge between $0\omega^{(n)}$ and $1\omega^{(n)}$, the argument is complete. □

Remark 1.2. *Readers with a* horror vacui *are advised to base induction on $n = 1$.*

Remark 1.3. *Combining the move types we get the more formal definition of the graphs R^n by*

$$V(R^n) = B^n,\ E(R^n) = \left\{\{\underline{s}0\omega^{(r-1)}, \underline{s}1\omega^{(r-1)}\} \mid r \in [n],\ \underline{s} \in B^{n-r}\right\},$$

where for each edge r is the moving ring. Note that the distribution $\underline{s}$ of rings $r+1$ to n is arbitrary.

Here is an alternative proof for Theorem 1.1. The path $P^n \subset R^n$ leading from $\alpha^{(n)}$ to $\omega^{(n)}$ must contain the edge $e := \{0\omega^{(n-1)}, 1\omega^{(n-1)}\}$, because this is the only legal way to move ring n onto the bar. This means that P^n contains, in obvious notation, $0P^{n-1}$, e, and $1P^{n-1}$, the latter traversed in inverse sense. Hence, $|P^n| \geq 2|P^{n-1}|$, such that, with $|P^0| = 1$, we get $|P^n| \geq 2^n$ and consequently $P^n = R^n$. In other words, R^n is obtained by taking two copies of R^{n-1}, reflecting the second, and joining them by an edge. As an example, R^3 is the path graph depicted in Figure 1.2. The reflection is indicated with the dashed line and the digits that were added to the graphs R^2 are in bold face.

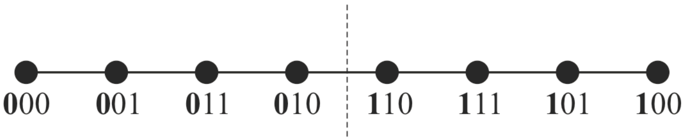

Figure 1.2: The graph R^3

From this it is obvious that the edge sets of the *Chinese rings graphs* can also be defined recursively.

Remark 1.4. *The edge sets of R^n are given by*

$$\begin{aligned} E(R^0) &= \varnothing, \\ \forall\, n \in \mathbb{N}_0 : E(R^{1+n}) &= \{\{ir, is\} \mid i \in B,\ \{r,s\} \in E(R^n)\} \cup \left\{\{0\omega^{(n)}, 1\omega^{(n)}\}\right\}. \end{aligned} \tag{1.1}$$

The reader is invited (Exercise 1.2) to deduce the recurrence

$$\beta_0 = 0, \ \forall\, n \in \mathbb{N}: \ \beta_n + \beta_{n-1} = 2^n - 1 \tag{1.2}$$

for the number β_n of moves needed to take off $n \in \mathbb{N}$ rings from the bar, i.e. to get from 1^n to 0^n in R^n. Obviously, $\beta = 0_2, 1_2, 10_2, 101_2, 1010_2, 10101_2, \ldots$ in binary representation, such that $\beta_n = \left\lceil \frac{2}{3}(2^n - 1) \right\rceil$ (A000975 in [296]; the sequence of differences $\beta_n - \beta_{n-1}$, $n \in \mathbb{N}$, is the Jacobsthal sequence [296, A001045]). The classical CR with $n = 9$ rings can therefore be solved in a minimum of 341 moves. In order to find out whether we have to move ring 1 or ring 2 first when we begin with all rings *on* the bar, we make the following observation.

Proposition 1.5. *The function* $B^n \ni s \mapsto \left(\sum_{r=1}^{n} s_r\right) \bmod 2 \in B$ *defines a vertex coloring of* R^n*; the type of the move associated with an edge defines an edge coloring.*

The proof is left as an exercise (Exercise 1.3).

Let us break to look back at early theories about the number of moves for a solution of the CR.

Since the first and then every second move on the path graph R^n is of type 0, ring 1 moves 2^{n-1} times on it. This makes up for 2^n Wallis movements (cf. p. 41). The other $2^{n-1} - 1$ moves produce $2^{n+1} - 4$ movements. For $n = 9$, this leads to 1532 movements; to this one has to add the movement of the bar right to the left end to arrive at Wallis's value in Section 0.6.1.

Another counting was used by Cardano, where a simultaneous move up or down of rings 1 and 2 is allowed for $n \geq 2$. Lucas called this the *accelerated Chinese rings* (ACR) in [209, p. 183–186]. The **diameter** of the corresponding graph reduces to $3 \cdot 2^{n-2} - 1$, and the standard task can then be solved in $\widetilde{\beta}_n = 2^{n-1} - (n \text{ even})$ moves; see Exercise 1.4.

It might be true that the Latin in Cardano's passage [50, p. 492f] is difficult to understand; possibly, as Gros [119] assumes, because written by a second hand. But it is clear from the beginning that he is dealing with ACR for $n = 7$. Therefore, the (only) numbers 31, 64, 95, and 190 occurring in the text are neither misprinted nor mathematical errors, as Gros and others claim, but are, respectively, $\widetilde{\beta}_6$, $\widetilde{\beta}_7$, $3 \cdot 2^5 - 1$, and twice the latter number. So Cardano gave the lengths of the solutions to get from 10^6 to 1^7, from 1^7 to 0^7, for the whole path and for the "circulus".

We are now also able[1] to interpret the two tables in the book by Zhu Xiang Zhuren mentioned in the introduction. We refer to Figure 0.24 whose contents have to be read from top to bottom and from right to left. The passage is entitled "Untangling Linked Rings Method" (解连环法). In the top row the rings are numbered above the puzzle (which has its handle on the right) as in Figure 1.1; instructions are given how to operate it and how to use the tables, the second of which "many times". The insert in the center row says "When possible, move the first two rings at the same time in one move.", that is, Cardano's counting is

[1] only with the generous aid by Wei Zhang and Peter Rasmussen

employed. Table I in the same row consists of 11 columns and 6 rows. The first case (top right) does not count, but has the overlaid inscription "This table takes the 9th ring off". And that is what it does! It starts (below the latter inscription) with "1 off, 3 off, 1 on, 2 and 1 off" and reaches the end of the first column with "4 off". (The Chinese characters for "on" (up) and "off" (down) are 上 (shang) and 下 (xia), respectively; the number symbols can easily be taken from the very top of the figure.) Then the second column starts with "2 and 1 on". Below the bottom left case, we find the inscription "The 9th ring is now off". In our notation this table solves the task $1^9 \to 010^7$, and the number of moves needed can be calculated from the dimensions of Table I as $11 \times 6 - 1 = 65$ and read from the line above the table. But for the last move, this is the solution for the 7-ring puzzle in $\widetilde{\beta}_7 = 64$ moves! The task is therefore reduced to solving $10^7 \to 0^8$, i.e. to traverse the whole path graph R^8, which should take $3 \cdot 2^6 - 1 = 191$ further moves. This is done with the aid of Table II in the bottom row of Figure 0.24. A first run leads to ring 8 being taken off, i.e. to state $0^2 10^6$, after $16 \times 6 = 96$ moves. (The entry in the penultimate case of the table wants to indicate the state *before* ring 8 is taken off; given that ring 9 was taken off by Table I, "rings 8 and 9 are on" is an obvious misprint for "rings 7 and 8 are on".) Instead of making use of the reflective structure of the state graph and going back Table II switching "on" and "off", the author now employs an iterative method, using the table once again from the beginning, but stopping in the middle, when ring 7 can be taken off. This is mathematically correct, because the first half of the CR graph is the same if a leading 0s is deleted. The kth application of Table II therefore solves the task $0^k 10^{8-k} \to 0^k 1^2 0^{7-k}$, i.e. $0^{8-k} \to 10^{7-k}$, in $3 \cdot 2^{6-k} - 1$ moves and then puts ring $9 - k$ down. So the 6th run ends already halfway through the rightmost column. This is indicated by the two lines on the margin saying that "rings 2 and 3 are on" and then "ring 3 is now off." Finally, the last two moves are in cases 4 and 5 of the right-hand column. The move numbers to get rings 3 to 8 off are again given above Table II. Adding up these values together with the two last moves gives the correct sum 191 indicated above.

Coming back to the standard style of counting individual ring moves, we have:

Proposition 1.6. *The CR with $n \in \mathbb{N}$ rings have a unique minimal solution of length $\left\lceil \frac{2}{3}(2^n - 1) \right\rceil$. It can be realized by alternating moves of types 0 and 1, starting with type 0 if n is odd and type 1 otherwise.*

Remark 1.7. *If one finds the CR abandoned in state $s \in B^n$, there is an easy way to decide about the best first move to get to $\alpha^{(n)}$ (the next moves being obvious on a path graph). As neighboring states differ by one bit only, the sum of bits of s will be odd if the last move on the path from $\alpha^{(n)}$ to s is made by ring 1 and even otherwise. So the parity of the sum of bits of s will tell you which move to make first on the reverse path.*

Every connected graph has a canonical metric, the **graph distance** $\mathrm{d}(s,t)$ between two vertices s and t being given by the length of a shortest s,t-path. A direct application is the existence of *perfect codes* for R^n, see Exercise 1.5. The proof of Theorem 1.1 shows that the diameter $\mathrm{diam}(R^n)$ of R^n is $\mathrm{d}(\alpha^{(n)},\omega^{(n)}) = 2^n - 1$ and that there is a unique shortest path between any two states of the CR. Their distance is given by $\mathrm{d}(s,t) = |\mathrm{d}(s) - \mathrm{d}(t)|$, where $\mathrm{d}(s) := \mathrm{d}(s,\alpha^{(n)})$ can be determined by a finite automaton (cf. Figure 1.3). It consists of two states A and B. The input of a bit a in A results in printing a and moving to state B if $a = 1$; the input of b in B leads to printing of $1 - b$ and moving to A if $b = 1$.

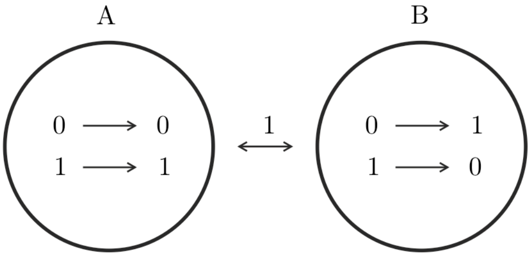

Figure 1.3: Automaton for the Gros code

Proposition 1.8. *If we enter the components* s_r *of* $s = s_n \dots s_1 \in B^n$, $n \in \mathbb{N}$, *from left to right, starting in state* A *of the automaton, the resulting output, read from left to right and interpreted as a binary number, gives the value of* $\mathrm{d}(s)$.

Proof. We prove by induction on n that input of s starting in A gives $\mathrm{d}(s)$ and input of s starting in B gives $2^n - 1 - \mathrm{d}(s)$. This is obviously true for $n = 1$. Let $n \in \mathbb{N}$ and $s \in B^{1+n}$. If $s = 0\overline{s}$, $\overline{s} \in B^n$, then starting in A leads to $\mathrm{d}(\overline{s}) = \mathrm{d}(s)$, because $\mathrm{d}(s) < 2^n$; starting in B gives $2^n + 2^n - 1 - \mathrm{d}(\overline{s}) = 2^{1+n} - 1 - \mathrm{d}(s)$. If $s = 1\overline{s}$, then starting in A leads to $2^n + 2^n - 1 - \mathrm{d}(\overline{s}) = \mathrm{d}(s)$, because $\mathrm{d}(s) = 2^n + \mathrm{d}(\overline{s},\omega^{(n)})$; starting in B gives $\mathrm{d}(\overline{s}) = 2^{1+n} - 1 - \mathrm{d}(s)$. □

Remark 1.9. *It follows immediately from Proposition 1.5 that the best first move from state s to state t is of type* $\mathrm{d}(s) \bmod 2$, *if* $\mathrm{d}(s) < \mathrm{d}(t)$ *and of type* $1-(\mathrm{d}(s) \bmod 2)$ *otherwise. The rest of the shortest path is then again obtained by alternating the types of moves.*

The bijection from B^n to $[2^n]_0$ provided by the automaton above is a coding of the states of the CR by the distance from the state $\alpha^{(n)}$. This code goes back to Louis Gros, who in 1872 published a theory of the *baguenaudier*, as the CR are called in French [119]. Its inverse is called *Gray code*, after F. Gray, who got a patent [116] for it in 1953. In fact, if d_r are the bits of $d \in \mathbb{N}_0$, we may put

$$\forall\, r \in \mathbb{N}:\; s_r = (d_r + d_{r-1}) \bmod 2\,. \tag{1.3}$$

Then $\mathrm{d}(s) = d$, as can be seen by applying the automaton or either Gros's formula $d_{r-1} = d_r + (1 - 2d_r)s_r$ (cf. [119, p. 13]). The corresponding automaton for the Gray code is shown in Figure 1.4. Here moves between the two states of the automaton are performed according to the one-sided arrows. For an application, see Exercise 1.6.

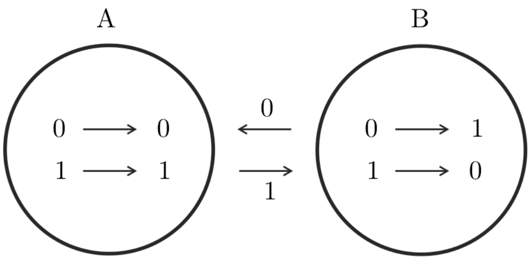

Figure 1.4: Automaton for the Gray code

Because of the construction in the proof of Theorem 1.1, the Gray code is also called *reflected binary code* (cf. Figure 1.2). Its main advantage is that neighboring code numbers differ by exactly one bit. More on Gray codes can be found in [180, Section 7.2.1.1]. For instance, it can also be produced in an iterative way with the aid of the Gros sequence; see Section 1.2 below.

A state $s \in B^n$ being uniquely determined by its distance $\mathrm{d}(s)$ from $\alpha^{(n)}$ and with all values from 0 to $2^n - 1$ occurring, it is obvious that the average distance to $\alpha^{(n)}$ (or to $\omega^{(n)}$ for that matter) in R^n is $2^{-n}\Delta_{2^n-1} = (2^n - 1)/2$. This is not too surprising given that R^n is isomorphic to the path graph on 2^n vertices P_{2^n}.

The **eccentricity** $\varepsilon(v)$ of a vertex v in a path graph is always the maximum of its distances to the two end vertices. Therefore, in R^{1+n}, $n \in \mathbb{N}_0$, we have

$$\forall\, i \in B,\ s \in B^n :\ 2^n \le \varepsilon(is) = 2^{n+1} - 1 - \mathrm{d}(s) \le 2^{n+1} - 1\,,$$

and every value in that range appears precisely once for each i. Hence, the **total eccentricity** of R^{1+n} is (cf. [296, A010036])

$$\mathrm{E}(R^{1+n}) = 2\sum_{k=2^n}^{2^{n+1}-1} k = (2^{n+1} - 1)2^{n+1} - (2^n - 1)2^n = 2^n(3 \cdot 2^n - 1)\,,$$

such that the *average eccentricity* on R^{1+n} turns out to be (cf. [154, Proposition 4.1])

$$\overline{\varepsilon}(R^{1+n}) = \frac{1}{2}(3 \cdot 2^n - 1)\,, \tag{1.4}$$

i.e. asymptotically (for $n \to \infty$) 3/4 of the diameter.

To find the average distance on R^n, we can start off from the notion of the *Wiener index* $W(G)$ of a connected graph G, namely the total sum

$W(G) = \sum_{\{s,t\} \in \binom{V(G)}{2}} \mathrm{d}(s,t)$ of distances between any two vertices of G. The *average distance* on G is then[2]

$$\overline{\mathrm{d}}(G) = \frac{2\,W(G)}{|V(G)|^2}\,.$$

Knowing the Wiener indices of path graphs from Exercise 1.7, we get

$$\overline{\mathrm{d}}(R^n) = \frac{1}{3}(2^n - 2^{-n})\,, \tag{1.5}$$

or approximately 1/3 of the diameter.

Finally, it would be interesting to know whether it is likely to arrive at a solution for the CR by chance, that is if the player makes moves at random choosing move types 0 or 1 with equal likelihood 1/2 (except in states $\alpha^{(n)}$ and $\omega^{(n)}$). Based on the method of Markov chains (cf. [203] and infra, Section 4.1), H. L. Wiesenberger has found [338, p. 66f] that the expected number of moves to reach state t from s on such a random walk in R^n is

$$\mathrm{d_e}(s,t) = \begin{cases} \mathrm{d}(t)^2 - \mathrm{d}(s)\,, & \text{if } \mathrm{d}(t) > \mathrm{d}(s)\,; \\ 0\,, & \text{if } \mathrm{d}(t) = \mathrm{d}(s)\,; \\ (2^n - 1 - \mathrm{d}(t))^2 - (2^n - 1 - \mathrm{d}(s))\,, & \text{if } \mathrm{d}(t) < \mathrm{d}(s)\,. \end{cases}$$

In particular, $\mathrm{d_e}(0^n, 1^n) = \beta_n^2$, but $\mathrm{d_e}(1^n, 0^n) = (2^n - 1)^2 - \beta_{n-1}$ for $n \in \mathbb{N}$. So it was a good idea of Hung Ming to give the jiulianhuan to his wife with all rings on the bar, because she would need 260951 moves to get them all off, in contrast to 116281 moves the other way round, if she wanted to solve it without thinking. These huge numbers demonstrate the advantage of a good mathematical model!

1.2 The Gros Sequence

CR yields an interesting integer sequence—the Gros sequence. It will become clear later that this is just the tip of an iceberg because many additional interesting integer sequences will appear in due course.

An analysis of the solution for the CR reveals the following: assuming that there are infinitely many rings and starting with all of them off the bar, the sequence of rings moved starts

$$g = 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, 1, 2, 1, 3, 1, \ldots$$

Note that we have paused at the moment when the first 5 rings are on the bar. This sequence is, of course, already anticipated in Zhu Xiang Zhuren's solution: the ring numbers in the bottom line of his Table II (cf. the bottom row in Figure 0.24) reading 4, 5, 4, 6, 4, 5, 4, 7, 4, 5, 4, 6, 4, 5, 4, 8. P. Rasmussen came up with a nice mnemonic:

[2] Some authors prefer to divide by $|V(G)|(|V(G)| - 1)$ only; cf. [46, Equation (9.4)].

one two one three one two one four one two one three one two one more .

To the best of our knowledge, the sequence g was studied for the first time by Gros in his 1872 pamphlet [119], hence we named it the *Gros sequence*. It is the sequence A001511 in [296] where it is referred to as the *ruler function*; cf. the markings on an imperial ruler, just like the heights of the columns in Figure 0.9. The Gros sequence obeys the following appealing recurrence.

Proposition 1.10. *For any* $k \in \mathbb{N}$,

$$g_k = \begin{cases} 1, & k \text{ odd;} \\ g_{k/2} + 1, & k \text{ even.} \end{cases}$$

Proof. Recall from Proposition 1.5 that in the solution of the CR, the moves alternate between type 0 and type 1. So every odd numbered move (or odd move for short) is of type 0 which implies that $g_k = 1$ when k is odd. Now consider even moves (that is, moves of rings $2, 3, 2, 4, \ldots$) and observe that they are identical to the solution of the puzzle with rings $2, 3, \ldots$. Since these moves appear in even steps, the second assertion follows. □

Repeated application of the recursion in Proposition 1.10 implies:

Corollary 1.11. *Let* $k \in \mathbb{N}$ *and write* $k = 2^r(2s+1)$, $r, s \geq 0$. *Then* $g_k = r + 1$.

Corollary 1.11 can be rephrased by saying that g_k is the number of times the factor 2 appears in $2k$. Already in 1808, A. M. Legendre posed the question, how many factors 2 the number $n!$ has [188, p. 8–10] (cf. [179, p. 51, Exercises 11/12]). This amounts to summing the *binary carry sequence* $\widetilde{g}$ [296, A007814], defined in Exercise 1.8, where it is shown that $\widetilde{g}_k = g_k - 1$. He obtained the remarkable formula

$$\sum_{k=1}^{n} \widetilde{g}_k = n - q(n), \tag{1.6}$$

where $q(n)$ denotes the number of 1s in the binary representation of n; see Exercise 1.9 for the proof.

Gros makes use of Corollary 1.11 for his "practical rule" [119, p. 14f] for a move in a state s with $\mathrm{d}(s) = k$: the move which led to s from $\alpha^{(n)}$ is by ring g_k, which is the position (counted from 1) of the rightmost bit 1 in k. Therefore, a move of this ring *from* s will lead to $\alpha^{(n)}$. By symmetry, moving away from $\alpha^{(n)}$ involves the ring with the position number of the rightmost 0 in the binary representation of k. Compared to our recipe in Remark 1.7, this procedure has the practical disadvantage that the state s is visible for the problem solver, but k has to be calculated. For the general task to get from s to t, solved by Lucas [214, p. 179], this is unavoidable though; cf. Remark 1.9.

Another use can be made of the binary representation to find out whether in move $k = \sum_{\nu=0}^{\infty} k_\nu \cdot 2^\nu$ ring $r = g_k$ goes up (1) onto the bar or down (0), still assuming an infinite supply of rings and starting with all of them off the bar. Let us denote the resulting binary sequence by $f \in B^{\mathbb{N}}$. Then $\forall\, k \in \mathbb{N}:\ f_k = 1 - k_{g_k}$, because this is the position (off or on the bar) of ring g_k after move number k according to (1.3). From Proposition 1.10 it is clear that f_k can be obtained recursively by

$$\forall\, \ell \in \mathbb{N}_0:\ f_{2\ell+1} = 1 - \ell \bmod 2,\ f_{2\ell+2} = f_{\ell+1}\,.$$

Let us look at the sequence f from a different prospect. Have you ever wondered why it is so difficult to fold a package insert of some medicine back after it had been unfolded? Try the inverse: take a lengthy strip of paper, fold it on the shorter center line and keep on doing this, always in the same direction, as long as it is physically feasible, the mth bending producing 2^{m-1} new folds. In the left picture of Figure 1.5 four foldings have been performed and colored according to their appearance, such that altogether 15 edges occur on the paper strip. Now unfold with approximately right angles at these fold edges. Despite the straightforward, symmetric procedure, the paper strip, viewed from the edge, will look surprisingly erratic as in the right-hand picture of Figure 1.5. Rotating at the center (red) fold, then at the secondary (green) one, then at blue and finally violet, you can solve the package insert problem.

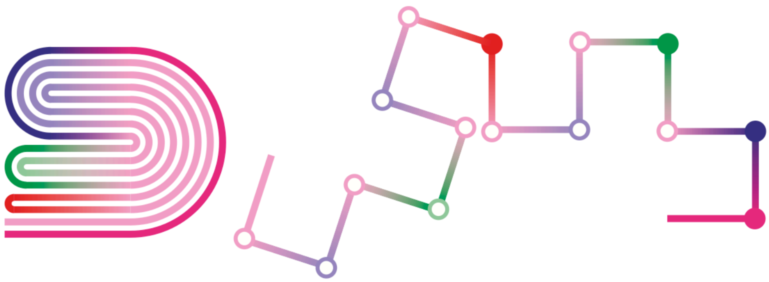

Figure 1.5: Folding and unfolding a paper strip

The arising polygon has been called a *dragon curve* (of order 4) because its shape tends (for higher orders) to the silhouette of a sea dragon; it is one of the favorites of fractal people (cf. [228, p. 66]) because of its interesting properties. For instance, in the top left picture of Figure 1.6 four copies of the curve of order 4 from Figure 1.5 meet at right angles in the center point. In the subsequent pictures the order grows up to 11, and at each step a scaling by the factor $1/\sqrt{2}$ has been performed such that the maximal squares inside which all gridpoints are covered have equal side-length. Figure 1.6 can be viewed as a proof without words for the space-filling property of dragon curves; cf. [276, p. 163f].

The folding is completely described by the sequence of orientations of the turns at the fold edges, i.e. the corners of the curve, either right (0) or left (1). For an $(m+1)$-fold folding, $m \in \mathbb{N}_0$, this *paper-folding sequence* $\varphi \in B^{\mathbb{N}}$ fulfills, if

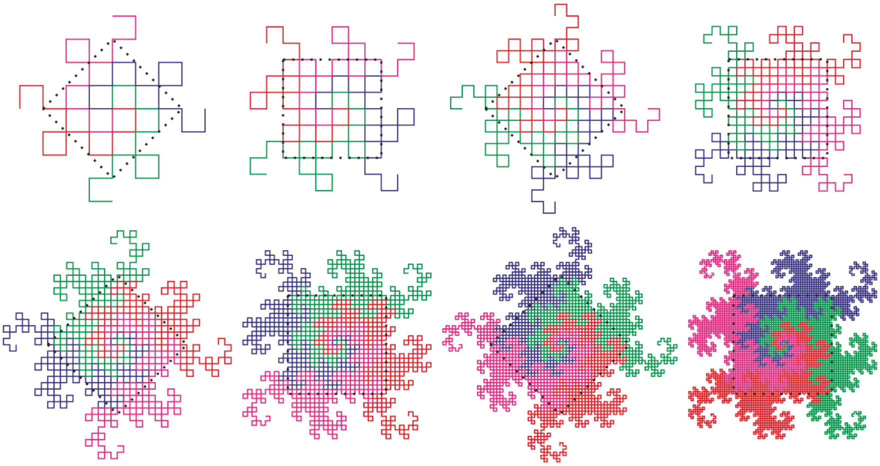

Figure 1.6: Space-filling property of the dragon curve

we start with a left bending,

$$\forall\, m \in \mathbb{N}_0 :\ \varphi_{2^m} = 1,\ \forall\, \mu \in [2^m - 1] :\ \varphi_{2^m+\mu} = 1 - \varphi_{2^m-\mu}$$

because of the reflection at the center fold 2^m. This is similar to the behavior of the Gros sequence as noticed by A. Sainte-Laguë [278, p. 40]. (The Gros sequence is misprinted there.) More precisely, as observed in [77], φ represents the pattern of ups and downs in the CR! Ring $r \in \mathbb{N}$ is moved up for the first time in move number 2^{r-1} after a sequence of moves leading from $\alpha^{(r-1)}$ to $\omega^{(r-1)}$ has been performed. This is followed by a complete transformation from $\omega^{(r-1)}$ to $\alpha^{(r-1)}$, i.e. the original subsequence in reflected order. So the sequence

$$f = 1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0, \ldots$$

is the paper-folding sequence φ (cf [296, A014577]), and our Chinese lady may amuse herself with either the CR or the folding of paper strips.

The Gros sequence can be found all over mathematics, for instance in connection with hamiltonian cycles on the edges of n-dimensional cubes; see Exercise 1.10. If in Figure 0.3 we start on bottom at 0 and change bits going around counter-clockwise according to the Gros sequence, i.e. $1, 2, 1, 3, 1, 2, 1$, we obtain the numbers from 0 to 7 in the order of the Gray code as in Figure 1.7.

Before we turn to some applications of the CR in Section 1.3, we show the presence of the Gros sequence in the fascinating theory of square-free sequences.

The greedy square-free sequence

A sequence $a = (a_n)_{n\in\mathbb{N}}$ of symbols a_n from an *alphabet* A is called *non-repetitive* or *square-free* (over A) if it does not contain a subsequence of the form xx (a

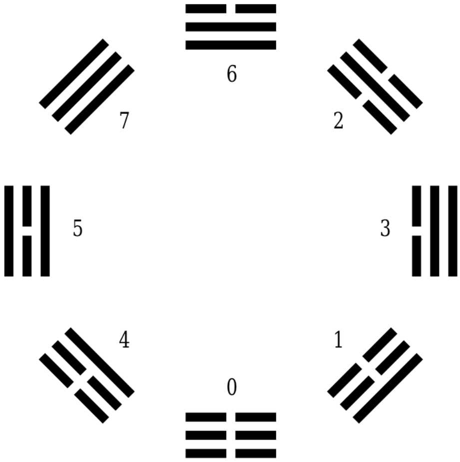

Figure 1.7: Gray's arrangement of the trigrams

square), where x is a non-empty subsequence of consecutive symbols of a. Clearly, $1, 2, 3, 4, \ldots$ is a square-free sequence over the alphabet $\mathbb{N}$. It is not very exciting though and expensive in the sense that it uses large numbers at early stages. So let us try to find a cheaper one by one of the most popular strategies in the theory of algorithms, namely the greedy approach; cf. p. 33. Roughly speaking, given a problem, a solution to the problem is built step by step, where in each step a partial solution is selected that optimizes certain greediness criteria.

In our case, the most obvious greediness criterion is to select the next term of the sequence as the smallest integer that does not produce a square. Let $a = (a_n)_{n\in\mathbb{N}}$ be the sequence obtained by this procedure; i.e.,

$$\forall\, n \in \mathbb{N}:\ a_n = \min\{\alpha \in \mathbb{N} \mid a_1, \ldots, a_{n-1}, \alpha \text{ contains no square}\}.$$

The sequence a is square-free, because any square would already occur at some finite stage of its construction. Clearly, $a_1 = 1$, $a_2 = 2$, and $a_3 = 1$. Then $a_4 \neq 1$ and $a_4 \neq 2$, hence $a_4 = 3$. Continuing such a reasoning it is easy to see that the sequence a begins as follows:

$$1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, 1, \ldots$$

From here we can guess the following result, a formal proof of which is left for Exercise 1.11:

Proposition 1.12. *The sequences g and a are the same.*

In particular, the Gros sequence is square-free ([144, Theorem 0]); it is even strongly square-free; cf. Exercise 1.12. We will return to square-free sequences in Chapter 2.

1.3 Two Applications

In this section we demonstrate how the theory of the CR and the Gros sequence can be helpful in the analysis of other problems.

Topological variations

We may turn our viewpoint around and ask whether there are other puzzles whose state graphs are isomorphic to R^n. In psychology literature they are called *problem isomorphs*. It has been regarded as a deficiency that the classical realization of R^9 in form of the CR as shown in Figure 1.1 physically allows us to move rings 1 and 2 simultaneously if they are both either on or off the bar. This is why we insisted on individual ring moves and counted them accordingly (see, however, Exercise 1.4). A variant which does not display this ambiguity was patented under the title “Locking disc puzzle” (LD) by W. Keister (cf. [164]). Here the intertwined system of rings of the CR is replaced by an arrangement of circular discs on a slide with three of the 90° sectors of the circle cut out to allow for rotation about their center (see Figure 1.8 for the 6-disc version). (To avoid a trivial solution, the first disc has only two sectors cut out and the one opposite to the convex side just flattened.) The bar of the CR corresponds to a frame in LD which is designed in a way to

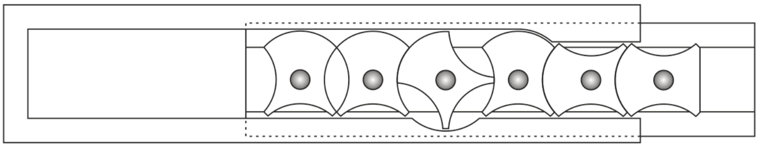

Figure 1.8: The Locking disc puzzle

realize the same kind of individual moves (i.e. rotation of discs and moving the slide back and forth) as for the CR. Thus the conditions “off” and “on” the bar of the rings in CR is translated into an orientation of the convex side of a disc parallel to the slide or perpendicular to it: as soon as all discs are positioned horizontally, the slide can be pulled out of the frame. However, this puzzle is *not* equivalent to the CR, because there are two horizontal orientations of the discs (a second vertical orientation is debarred by the frame), such that the corresponding state graph has more vertices than R^6 and therefore can not be isomorphic to it.

In a second attempt to improve the CR hardware, Keister came forward with the “Pattern-matching puzzle” (PM) (cf. [165]), where in addition to a slide and a frame there is a rack attached to the frame which holds 4 pattern bars. The latter constitute a 4-bit code which is fixed at the beginning. Due to the mechanical arrangement of PM only one of the 8 teeter bars on the slide can be moved at a time and this move of bar $b \in [8]$ between two positions “up” (1) and “down” (0) is possible only if the bars with smaller labels coincide with the given code (filled up to the right by 0s; we adapted the labellings to our CR notations). Again the task starts with all teeter bars up, i.e. in state 1^8, and the slide can be detached from the frame only in state 0^8. It turns out that one can reach each of the 2^8 states of the teeter bars from every other one in a unique path, such that the correspnding state graph for each of the 16 codings of the pattern bars is a tree on 2^8 vertices. Therefore, the task to free the slide has different (lengths of) solutions depending on the code, 0000 leading to the shortest one with just 8 moves. Only in the special case 1000 of the pattern code the corresponding tree is actually a path graph and the puzzle isomorphic to the CR with 8 rings, such that the task needs $\beta_8 = 10101010_2 = 170$ moves.

A much more challenging variant of the CR is depicted in Figure 1.9, and following E. R. Berlekamp, J. H. Conway and R. K. Guy [34, p. 858] we will call it the *Chinese string (CS)*. Here the system of $n \in \mathbb{N}$ rings in the original puzzle is arranged in a rigid, but otherwise topologically equivalent manner in a *frame*, e.g., made of wood. The shuttle of the CR is replaced by a flexible *rope* (or string) of sufficient length. At the beginning, the rope, initially separated from the frame, is somehow entangled with it and the task for the player is to disentangle the rope from the frame. (The dotted arc in Figure 1.9 does not belong to the puzzle and will be explained later.)

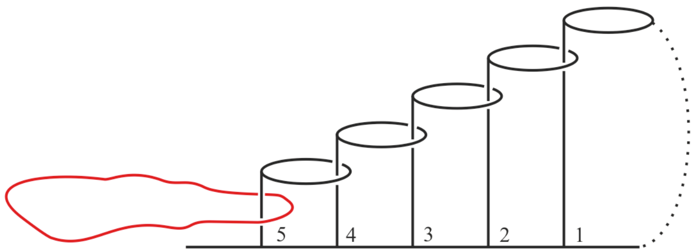

Figure 1.9: Chinese string puzzle

Because of the flexibility of the rope it is clear that there are many more possible states of the CS puzzle, but that any distribution $s \in B^n$ of the CR can also be realized in CS by inserting the rope into the frame in the same geometric fashion as the shuttle is moved into the rings in CR. For an example, see Figure 1.10 showing $s = 11001$.

Since there is no obvious notion of a *move* in CS, the complexity of a solution has to be defined in a topological way. In [163], L. H. Kauffman suggests to add an imaginary arc running from the tip of ring 1 to the base of the frame as in Figure 1.9 and to count the number of crossings of this arc with the loop during a solution; the minimal number of crossings taken over all solutions is called the *exchange number* of s. Every solution to get from s to $\alpha^{(n)}$ in CR can also be performed on CS. Herein only the moves of ring 1 will cross the arc. Therefore the exchange number is bounded from above by the number of moves of ring 1 in the

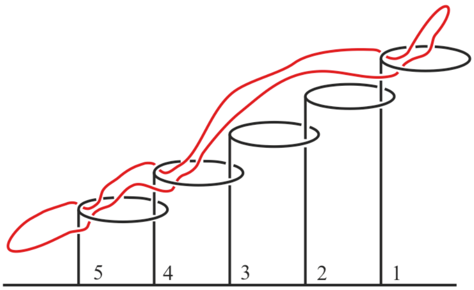

Figure 1.10: State 11001 of CS

optimal solution in CR, which is $\left\lceil \frac{\mathrm{d}(s)}{2} \right\rceil$. Kauffman's *Ring conjecture* [163, p. 8] says that this is also a lower bound for the exchange number of s. This conjecture has been confirmed by J. H. Przytycki and A. S. Sikora in [266, Theorem 1.1] for the special case $s = \omega^{(n)}$ (cf. Figure 1.9 for $n = 5$), where the exchange number is 2^{n-1}. The general case seems to be open.

For psychological tests it might be interesting to compare the performance of subjects who are first confronted with the CR and thereafter with the CS or vice versa, because it seems to be very confusing that fixed and moving parts of the puzzles are interchanged in the two versions.

Tower of Hanoi networks

We next briefly describe an application of the sequence g from physics. S. Boettcher, B. Gonçalves, and H. Guclu [39, 40] introduced two infinite graphs named *Tower of Hanoi networks*. They introduced them to explore aspects of a small-world behavior and demonstrated that they possess appealing properties.

The network/graph HN4 is defined on the vertex set $\mathbb{Z}$. Connect each vertex k to $k+1$ and $k-1$, so that the two-way infinite path is constructed. Next write $k \in \mathbb{Z} \setminus \{0\}$ as $k = 2^r(2s+1)$, $r \in \mathbb{N}_0$, $s \in \mathbb{Z}$, and connect k to $2^r(2s+3)$, and $2^r(2s-1)$. Finally add a loop at the vertex 0 so that the constructed graph HN4 is 4-regular. It is shown in Figure 1.11.

The graph HN3 is defined similarly, except that it is built on the basis of the one-way infinite path to which only edges that form "forward jumps" are added. Instead of giving its formal definition we refer to Figure 1.12.

To see the connection between the networks HN3/HN4 and the sequence g, consider the vertices of HN4 that belong to $\mathbb{N}$. By the definition of the network, a vertex $k = 2^r(2s+1)$ has four neighbors, the largest of them being $2^r(2s+3)$. The jump from k to $2^r(2s+3)$ is

$$2^r(2s+3) - 2^r(2s+1) = 2^{r+1} = 2^{g_k},$$

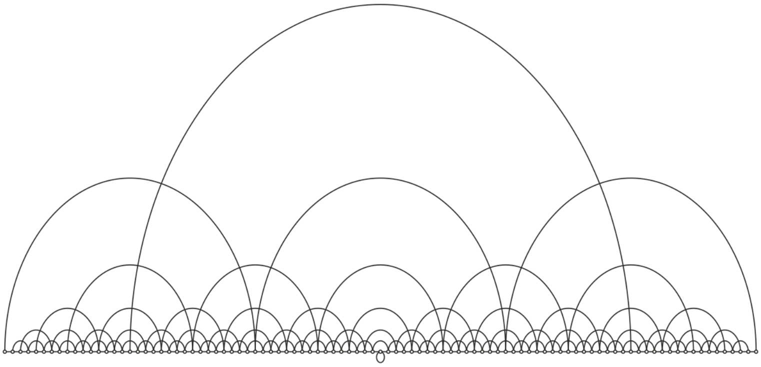

Figure 1.11: Network HN4

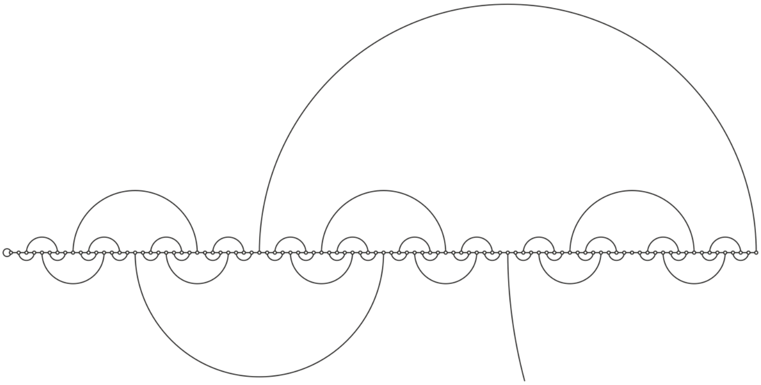

Figure 1.12: Network HN3

the latter equality following from Corollary 1.11. A similar conclusion holds for HN3. Hence a historically better justified name for the HN3/HN4 networks would be *Chinese rings networks.*

As already mentioned, the sequence g appears in many situations, too many to be even listed here. The biggest surprise, however, is that the Gros sequence solved the Tower of Hanoi before the latter was at all invented!

1.4 Exercises

1.1. Show that a graph with all vertices of degree 2 except for two pendant vertices is the union of a path with some (maybe no) cycles.

1.2. Derive recurrence (1.2).

1.3. Prove Proposition 1.5.

1.4. Consider the ACR for $n \geq 2$.

a) What is the corresponding state graph?

b) Calculate the value of its diameter.

c) Determine the minimal number of moves $\widetilde{\beta}_n$ needed to solve the ACR task $1^n \to 0^n$ (or vice versa).

1.5. A *perfect code* in a connected graph $G = (V, E)$ is a subset C of the vertex set V with the property

$$\forall\, v \in V\ \exists_1\, c \in C:\ \mathrm{d}(v, c) \leq 1\,.$$

The elements of C are called *codewords.*

Show that R^n, $n \in \mathbb{N}_0$, contains precisely two perfect codes C, if n is odd, and precisely one, if n is even. How large is $|C|$?

1.6. [79, Problem 417] Suppose there are altogether fourteen rings on the tiring irons [Chinese rings] and we proceed to take them all off in the correct way so as not to waste any moves. What will be the position of the rings after the 9999th move has been made?

1.7. Determine the Wiener index of the path graph on $k \in \mathbb{N}$ vertices P_k.

1.8. Let $k = (b_n b_{n-1} \ldots b_1 b_0)_2$ be the binary representation of an integer $k \in \mathbb{N}$. Define $\widetilde{g}_k = i$, where $b_i = 1$ and $b_j = 0$ for $j \in [i]_0$. (That is, $\widetilde{g}_k$ is the index of the right-most non-zero bit in the binary representation of k. For instance, $\widetilde{g}_1 = 0$ and $\widetilde{g}_{24} = 3$.) Show that

$$\widetilde{g}_k + 1 = g_k\,.$$

1.9. Show that for $k \in \mathbb{N}$, the function q fulfills

$$q(k) = \begin{cases} q(k-1) + 1, & k \text{ odd;} \\ q\left(\frac{k}{2}\right), & k \text{ even} \end{cases}$$

and deduce Legendre's formula (1.6) from this.

1.10. The n-cube is the graph with vertex set $\{b_1 b_2 \ldots b_n \mid b_i \in B\}$, two vertices being adjacent if their labels differ in exactly one position. Show that the n-cube contains a hamiltonian cycle for any $n \geq 2$.

1.11. Show that the Gros sequence and the greedy square-free sequence are the same.

1.12. [144, p. 259] A sequence of symbols $a = (a_n)_{n \in \mathbb{N}}$ is called *strongly square-free* if it does not contain a subsequence of the form xy (an *abelian square*), where x and y are non-empty subsequences of consecutive symbols of a such that the symbols from y form a permutation of the symbols from x. Show that the Gros sequence is strongly square-free.