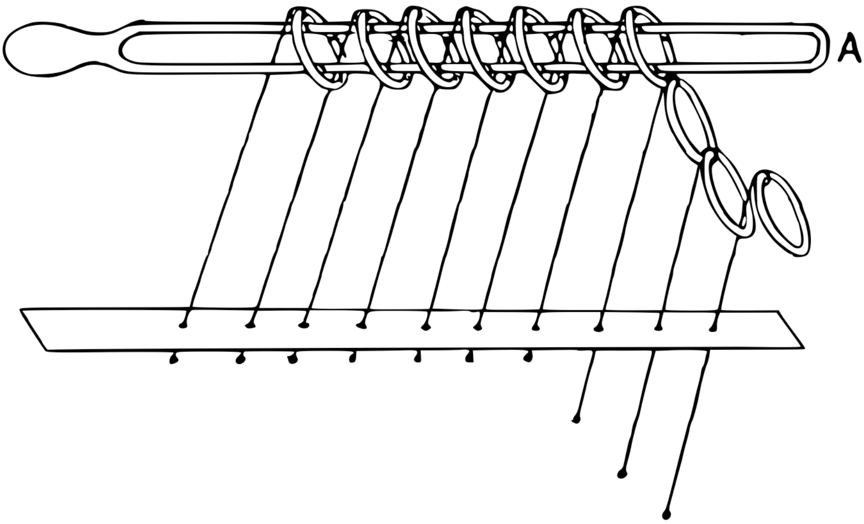
Mathematical Recreations and Problems of Past and Present Times
W.W. Rouse Ball: 1892/1905

Chinese Rings*. A somewhat more elaborate toy, known as *Chinese Rings*, which is on sale in most English toy-shops, is represented in the accompanying figure. It consists of a number of rings



hung upon a bar in such a manner that the ring at one end (say A) can be taken off or put on the bar at pleasure; but any other ring can be taken off or put on only when the one next to it towards A is on, and all the rest towards A are off the bar. The order of the rings cannot be changed.

Only one ring can be taken off or put on at a time. [In the toy, as usually sold, the first two rings form an exception to the rule. Both these can be taken off or put on together. To simplify the discussion I shall assume at first that only one ring is taken off or put on at a time.] I proceed to show that, if there are n rings, then in order to

* It was described by Cardan in 1550 in his *De Subtilitate*, bk. XV, paragr. 2, ed. Sponius, vol. III, p. 587; by Wallis in his *Algebra*, second edition, 1693, *Opera*, vol. II, chap. 111, pp. 472–478; and allusion is made to it also in Ozanam's *Récréations*, 1723 edition, vol. IV, p. 439.

disconnect them from the bar, it will be necessary to take a ring off or to put a ring on either $\frac{1}{3}(2^{n+1}-1)$ times or $\frac{1}{3}(2^{n+1}-2)$ times according as n is odd or even.

Let the taking a ring off the bar or putting a ring on the bar be called a *step*. It is usual to number the rings from the free end A. Let us suppose that we commence with the first m rings off the bar and all the rest on the bar; and suppose that then it requires $x-1$ steps to take off the next ring, that is, it requires $x-1$ additional steps to arrange the rings so that the first $m+1$ of them are off the bar and all the rest are on it. Before taking these steps we can take off the $(m+2)$th ring and thus it will require x steps from our initial position to remove the $(m+1)$th and $(m+2)$th rings.

Suppose that these x steps have been made and that thus the first $m+2$ rings are off the bar and the rest on it, and let us find how many additional steps are now necessary to take off the $(m+3)$th and $(m+4)$th rings. To take these off we begin by taking off the $(m+4)$th ring: this requires 1 step. Before we can take off the $(m+3)$th we must arrange the rings so that the $(m+2)$th is on and the first $m+1$ rings are off: to effect this, (i) we must get the $(m+1)$th ring on and the first m rings off, which requires $x-1$ steps, (ii) then we must put on the $(m+2)$th ring, which requires 1 step, (iii) and lastly we must take the $(m+1)$th ring off, which requires $x-1$ steps: thus this series of movements requires in all $\{2(x-1)+1\}$ steps. Next we can take the $(m+3)$th ring off, which requires 1 step; this leaves us with the first $m+1$ rings off, the $(m+2)$th on, the $(m+3)$th off and all the rest on. Finally to take off the $(m+2)$th ring, (i) we get the $(m+1)$th ring on and the first m rings off, which requires $x-1$ steps, (ii) we take off the $(m+2)$th ring, which requires 1 step, (iii) we take $(m+1)$th ring off, which requires $x-1$ steps: thus this series of movements requires $\{2(x-1)+1\}$ steps.

Therefore, if when the first m rings are off it requires x steps to take off the $(m+1)$th and $(m+2)$th rings, then the number of additional steps required to take off the $(m+3)$th and $(m+4)$th rings is $1+\{2(x-1)+1\}+1+\{2(x-1)+1\}$, that is, is $4x$.

To find the whole number of steps necessary to take off an odd number of rings we proceed as follows.

To take off the first ring requires 1 step;
∴ to take off the first 3 rings requires 4 additional steps;
∴ " " 5 " " 4^2 " " .

In this way we see that the number of steps required to take off the first $2n+1$ rings is $1+4+4^2+\cdots+4^n$, which is equal to $\frac{1}{3}(2^{2n+2}-1)$.

To find the number of steps necessary to take off an even number of rings we proceed in a similar manner.

To take off the first 2 rings requires 2 steps;
∴ to take off the first 4 rings requires 2×4 additional steps;
∴ " " 6 " " 2×4^2 " " .

In this way we see that the number of steps required to take off the first $2n$ rings is $2+(2\times 4)+(2\times 4^2)+\cdots+(2\times 4^{n-1})$, which is equal to $\frac{1}{3}(2^{2n+1}-2)$.

If we take off or put on the first two rings in one step instead of two separate steps, these results become respectively 2^{2n} and $2^{2n-1}-1$.

I give the above analysis because it is the direct solution of a problem attacked by Cardan in 1550 and by Wallis in 1693—in both cases unsuccessfully—and which at one time attracted some attention. I proceed next to give another solution, more elegant though rather artificial.

This solution, which is due to M. Gros[*], depends on a convention by which any position of the rings is denoted by a certain number expressed in the binary scale of notation in such a way that a step is indicated by the addition or subtraction of unity.

Let the rings be indicated by circles: if a ring is on the bar, it is represented by a circle drawn above the bar; if the ring is off the bar, it is represented by a circle below the bar. Thus figure i below represents a set of seven rings of which the first two are off the bar, the next three are on it, the sixth is off it, and the seventh is on it.

Denote the rings which are on the bar by the digits 1 or 0 alternately, reckoning from left to right, and denote a ring which is off the bar by the digit assigned to that ring on the bar which is nearest to it on the left of it, or by a 0 if there is no ring to the left of it.

Thus the three positions indicated below are denoted respectively by the numbers written below them. The position represented in figure ii is obtained from that in figure i by putting the first ring on to the bar, while the position represented in figure iii is obtained from that in figure i by taking the fourth ring off the bar.

It follows that every position of the rings is denoted by a number expressed in the binary scale: moreover, since in going from left to right

* *Théorie du Baguenodier*, by L. Gros, Lyons, 1872. I take the account of this from Lucas, vol. I, part 7.

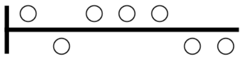


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Figure i.

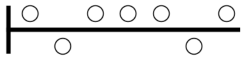

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Figure ii.

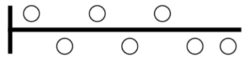

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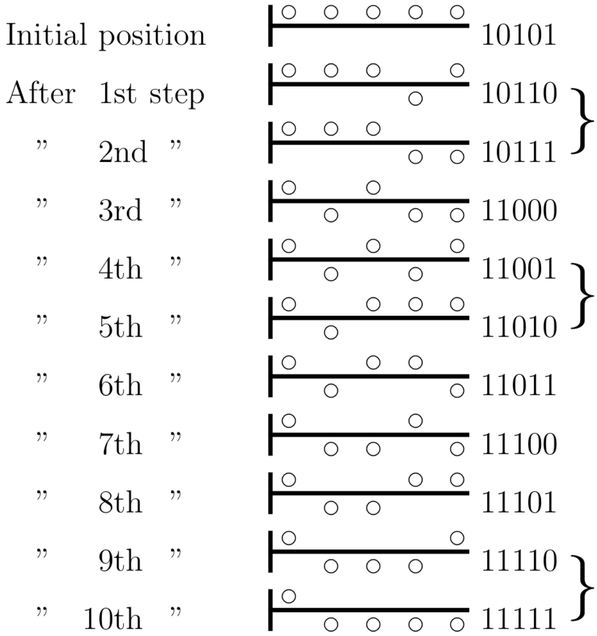
Figure iii.

every ring on the bar gives a variation (that is, 1 to 0 or 0 to 1) and every ring off the bar gives a continuation, the effect of a step by which a ring is taken off or put on the bar is either to subtract unity from this number or to add unity to it. For example, the number denoting the position of the rings in figure ii is obtained from the number denoting that in figure i by adding unity to it. Similarly the number denoting the position of the rings in figure iii is obtained from the number denoting that in figure i by subtracting unity from it.

The position when all the seven rings are off the bar is denoted by the number 0000000: when all of them are on the bar by the number 1010101. Hence to change from one position to the other requires a number of steps equal to the difference between these two numbers in the binary scale. The first of these numbers is 0: the second is equal to $2^6 + 2^4 + 2^2 + 1$, that is, to 85. Therefore 85 steps are required. In a similar way we may show that to put on a set of $2n + 1$ rings requires $(1 + 2^1 + 2^2 + \ldots + 2^{2n})$ steps, that is, $\frac{1}{3}(2^{2n+2} - 1)$ steps; and to put on a set of $2n$ rings requires $(2 + 2^3 + \ldots + 2^{2n-1})$ steps, that is, $\frac{1}{3}(2^{n+1} - 2)$ steps.

I append a table indicating the steps necessary to take off the first four rings from a set of five rings. The diagrams in the middle column show the successive position of the rings after each step. The number following each diagram indicates that position, each number being obtained from the one above it by the addition of unity. The steps which are bracketed together can be made in one movement, and, if thus effected, the whole process is completed in 7 movements instead of 10 steps: this is in accordance with the formula given above.

M. Gros asserted that it is possible to take from 64 to 80 steps a minute, which in my experience is a rather high estimate. If we accept the lower of these numbers, it would be possible to take off 10 rings in less than 8 minutes; to take off 25 rings would require more than 582 days, each of ten hours work; and to take off 60 rings would



necessitate no less than 768614, 336404, 564650 steps, and would require nearly 55000, 000000 years work—assuming of course that no mistakes were made.