Dürer's Magic Square, Cardano's Rings, Prince Rupert's Cube, and Other Neat Things

V. Frederick Rickey

Abstract: Recreational mathematics is as old as mathematics itself, so a survey of its history is out of the question. Instead we discuss a few neat things, setting each in its historical context and explaining their significance. As a benchmark for looking forward and back we shall take Charles Hutton's *Recreations in Mathematics and Natural Philosophy*, which in turn is based on works of Ozanam and Montucla on recreational mathematics.

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Cardano's Rings

Perhaps the oldest mathematical recreation is the Chinese Rings puzzle or Cardano's Rings as I shall call them. The puzzle consists of seven or ten or whatever number of rings, each threaded through the eye of a short post that is able to slide up and down in a solid base. Connecting the rings is a long oval or slotted bar that goes through them and over the posts. The goal is to manipulate the rings and free the bar from the rest of the device. The rings, posts, and base are permanently hooked together but there is considerable room for movement as the posts can slide up and down through the base and the rings can slide around the eyes in the posts. It is sure to provide you with several hours of mathematical recreation.

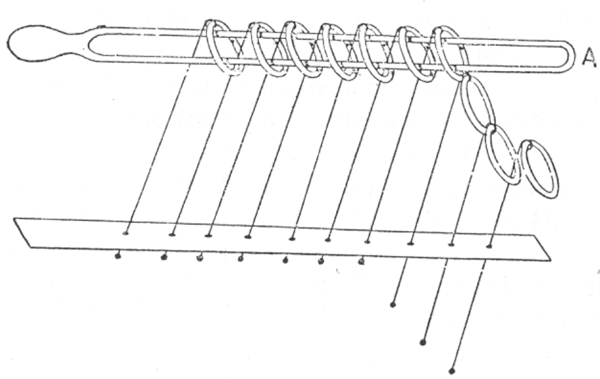

Figure 1. Cardano's Rings

According to the ethnographer Stewart Culin (1858–1929) this puzzle was known in the Chinese Sung Dynasty (960–1279). His *Games of the Orient: Korea, China, Japan* (1895 with another title; reprinted 1959 and 1991 under this title) attributes the puzzle to Hung Ming (181–234), a famous Chinese hero. I am skeptical of this early origin and have been unable to track down a solid modern reference to such an early date. Another puzzle of supposedly ancient Chinese roots is the Tower of Hanoi puzzle. It was published by Édouard Lucas (1842–1891) in his four volume work, *Récréations mathématiques* (1883) under the name 'M. Claus,' an anagram of Lucas. The next year the scientist Henri de Parville (1838–1909) made up the story about the monks of Siam moving 64 gold disks between three diamond needles. The world was to end when the task was completed [36, p. 304]. While that has not yet happened, the world of Lucas ended in an unusual way. At a banquet he was attending, a waiter dropped some dinnerware and Lucas was cut on the cheek with a shard. A few days later he died of septicemia.

The earliest mention of this puzzle in print, where it is called a 'meleda,' is in Girolamo Cardano's *De subtilitate libri XXI* (A Treatise on Subtelty, 1550; book XV, paragraph 2), an encyclopedia of wide range which contains sound subjects ranging from natural philosophy, cosmology, mechanics and cryptography, and the construction of machines to such unsound subjects as alchemy, the occult, and the evil influence of demons. This work was widely read, there being five editions in the remainder of the sixteenth century [39]. Cardano is best known to mathematicians as author of the *Ars Magna* (1545)[38, 41], a book that explains how to solve third and fourth degree polynomial equations.

Recently, however, a manuscript of Luca Pacioli (1445–1517) has turned up that contains a description of the Chinese Rings [42]. This places their European origin around 1500.

After Cardano, the next mention of the puzzle is not in the 1685 English edition of John Wallis's *Algebra* [46], as is often incorrectly said, but in the Latin edition of 1693 [49], where it is described in Caput CXI (pp. 472–478), which is entitled 'De complicatis annulis,' the complicated rings. Wallis gives a description of how to solve the puzzle and it is for this reason that I refer to it as 'Cardano's Rings.'

Ball and Coxeter have a footnote [36, p. 305] indicating that Cardano's Rings are pictured without explanation in the *Récreations Mathématiques* in the 1723 edition (volume 4, p. 439), but I have not seen this edition. There is no mention of the rings in Hutton's English translation of this work. Nonetheless, we include them here because they are part of the Ozanam-Hutton tradition.

One sign of the popularity of this puzzle is that it has been patented at least 34 times in half-a-dozen countries in the twentieth-century, including twenty-one times in the United States [42] and is still sold under a number of names. When traveling to the midwest, my wife and I occasionally stop at Yoder's Shopping Center in Shipshewana, Indiana, an Amish area of the state. While my wife looks at fabric, I look first for material for neckties, and then at the wide variety of tools and household goods that this old-fashioned shop has (if anyone needs a lamp wick, a hand-cranked five-quart ice cream maker, or replacement blades for a meat grinder, this is the place to look: http://www.yodershardware.com). On a trip earlier this summer, I was pleased to be able to purchase a set of Cardano's Rings, or a 'patience puzzle,' as it was called. This is one of the twenty puzzles in the 'Tavern Puzzle Collection' manufactured by the blacksmith Dennis Sucilsky (http://www.tavernpuzzle.com).

I encourage you to find a copy of Cardano's Rings and spend a few dozen hours trying to solve the puzzle. If you become frustrated you can buy one from blacksmith Sucilsky (they come with solutions), consult [37], or read on. But if you want to try to master it yourself, skip the remainder of this section.

W. W. Rouse Ball, in his delightful book on mathematical recreations gives a nice presentation of the solution to this puzzle. First he describes the puzzle and gives the key to its solution:

> It consists of a number of rings hung upon a bar in such a manner that the ring at one end (say A) can be taken off or put on the bar at pleasure; but any other ring can be taken off or put on only when the one next to it towards A is on, and all the rest towards A are off the bar. The order of the rings cannot be changed. [36, p. 305]

Using straightforward counting techniques, Ball develops a recursive relationship for the solution, and shows that if the number n of rings is odd then $(2^{n+1} - 1)/3$ steps are needed whereas $(2^{n+1} - 2)/3$ are needed if n is even. A 'step' is when a ring is put on or taken off the bar.

Then he gives "another solution, more elegant, though rather artificial." Ball attributes this to L. Gros, *Théorie du Baguenodier (Theory of the Time-Waster)* (1872). What is artificial about this solution is the way binary numerals are assigned to the different arrangements of the rings:

> Denote the rings which are on the bar by the digits 1 or 0 alternately, reckoning from left to right, and denote a ring which is on the bar by the digit assigned to that ring on the bar which is nearest to it on the left of it, or by 0 if there is no ring to the left of it. [36, p. 308].

When the 10 rings of the exemplar in Figure 1 are all on the bar, then the Gros code is 1010101010. When they are all off the bar, the code is 0000000000. In Figure 1, the arrangement is 1010101111. With a little practice it is easy to convert back and forth between the rings and the Gros coding. Now solution of the puzzle is particularly easy with this notation. Start with 1010101010 as the initial position and subtract 1 (using binary subtraction). This is the next position. Now repeatedly subtract 1 until you get to a string of 0s; these Gros codes give the intermediate positions. To see how many moves this will take, subtract the ending position from the starting position and convert to decimal; in this case it takes 682 moves. To put the rings back on the bar, repeatedly add 1. Of course this method works if someone has left your puzzle partway worked and you want to put it back in the original shape; you can even compute how many moves it will take in advance by doing the binary subtraction. Ball was right, the coding is artificial, but the solution is elegant.

There is a second way of coding that also solves the ring puzzle. If a ring is on the bar denote it by 1, if off by 0. This assigns a string of 0s and 1s to every position of the puzzle. Now in solving the puzzle only one ring can move on or off the bar at a time. The codes for those two positions differ in only one digit. This prompts us to define a *Gray code* as an ordering of the 2^n sequences of length n consisting of 0s and 1s in such a way that adjacent sequences differ by just one digit. Such sequences were patented by Frank Gray in 1953 (U.S. patent 2 632 058); hence the name. These Gray codes are never unique and it is still an open question as to how many of them there are for a given n [40].

Now not any Gray sequence will provide a solution to Cardano's Ring puzzle, because some adjacent codes do not represent posssible moves on the puzzle. The only Gray code that will work is the reflected Gray code. Here is how to construct it. Begin with $G_1 = 0, 1$, i.e., the sequence of 0 followed by 1. Then let

$$G_{n+1} = 0G_n \cup 1G'_n$$

i.e., to go from G_n to G_{n+1} concatnenate 1 with each code in G_n and follow that sequence with 0 concatenated with each code in G_n but with the order reversed. Here are the first few codes:

$$\begin{aligned} G_1 &= 0, 1 \\ G_2 &= 00, 01, 11, 10 \\ G_3 &= 000, 001, 011, 010, 110, 111, 101, 100 \end{aligned}$$

Let us now give an example comparing the two codes and how they are used to solve the 4-ring puzzle. Note that although G_4 contains 16 elements, we only use those between 0000 and 1111:

Gray	Gros
1111	1010
1101	1001
1100	1000
0100	0111
0101	0110
0111	0101
0110	0100
0010	0011
0011	0010
0001	0001
0000	0000

Note that with the reflecting Gray code one must write out the code before one can use it to solve the puzzle. This time the coding is elegant, but the solution is artificial.

The line between recreational mathematics and the mathematics of the professional mathematician is never entirely clear. In a conversation about this paper earlier this summer, a former colleague Charles Holland remarked "Isn't all mathematics recreational mathematics?" While I am inclined to agree with him, this puzzle crossed the line between recreational and research by motivating a paper by Przytycki and Sikora that is published in the *Proceedings of the AMS* [43]. The paper uses low-dimensional topology and group theory to prove a conjecture of L. Kauffman that the minimum number of moves to solve Cardano's Rings with n-rings is 2^{n-1}, but they use a different method of counting moves.

Cardano's Rings

[36] Ball, W. W. Rouse (1850–1925), and Coxeter, H. S. M. (1907–2003) *Mathematical recreations & essays*. Toronto: University of Toronto Press, 12th edition, 1974. MR0351741 (50 #4229)

[37] Berlekamp, Elwyn R.; Conway, John Horton; and Guy, Richard K., *Winning Ways* (1982). See volume 2, p. 748.

[38] Cardano, Girolamo (1501–1576), *Artis magnae sive de regulis algebraicis liber unus*, 1545. English translation by T. Richard Witmer, *The Great Art or the Rules of Algebra by Girolamo Cardano*, Cambridge: MIT Press, 1968; Dover 1993. Reviewed MR0250842 (40 #4074) by J. Vernet.

[39] Fierz, Markus, *Girolamo Cardano, 1501–1576. Physician, Natural Philosopher, Mathematician, Astrologer, and Interpreter of Dreams*, Boston: Birkhäuser, 1983. Translated by Helga Niman from the German original with the same title.

The subtitle accurately describes the chapters of this book. Cardano's most read work was the encyclopedic work *De subtilitate* (A Treatise on Subtelty, 1550), a work of over 700 pages and double that in later editions. Cardano's cube is discussed in book 15, "De inutilibus subtilitatibus" (Useless subtleties), but no mention is made of this by Fierz, who does give a thumbnail sketch of the mathematics in the work.

[40] Gardner, Martin, "The binary Gray code," pp. 11–27 in *Knotted Doughnuts and Other Mathematical Entertainments*, New York: W. H. Freeman, 1986. An expanded version of the "Mathematical Games" column in *Scientific American*, vol. 227, no. 2, August 1972.

[41] Ore, Oystein, *Cardano, the Gambling Scholar*, Princeton, NJ: Princeton University Press, 1953.

[42] Pacioli, Luca (1445–1517), "Capitolo CVII. Do cavare et mettere una strenghetta salda in al quanti analli saldi difficil caso," http://www.uriland.it/mathematica/DeViribus/2_113.html .

[43] Przytycki, Jósef H. and Sikora, Adam S., "Topological insights from the Chinese rings," *Proceedings of the American Mathematical Society*, 130 (2002), 893–902.

[44] Scott, Joseph Frederick (1892–1971), *The Mathematical Work of John Wallis, D.D., F.R.S. (1616-1703)*, London, 1938. Reprinted by Chelsea, 1981.

[45] Scott, Joseph Frederick, "John Wallis as historian of mathematics," *Annals of Science*, 1 (1936), 335–357.

[46] Wallis, John (1616–1703), *A Treatise on Algebra both Historical and Practical Shewing the Original, Progress, and Advancement Thereof, from Time to Time; and by the Steps it Hath Attainted to the Height at Which now it is,* London, 1685.

[47] Wallis, John, *De sectionibus conicis, nova methodo expositis, tractatus*, pp. 49–108 of volume I of Wallis's *Operum mathematicorum* (2 volumes, Oxford, 1656-1657) and reprinted pp. 291–354 of volume I of Wallis's *Opera mathematica* (3 volumes, Oxford, 1693–1699). These three volumes were reprinted Hildesheim: Olms, 1972.

[48] Wallis, John, *Arithmetica infinitorum*, pp. 1-199 of volume II of Wallis's *Operum mathematicorum* and reprinted pp. 355-478 of volume I of Wallis's *Opera mathematica*.

[49] Wallis, John, *De algebra tractatus, historicus & practicus*, pp. 1–482 of volume II of Wallis's *Opera mathematica*, 1693

This is a Latin translation of his 1685 with the addition of 112 chapters. Available via Early English Books Online if your library subscribes.

[50] Wallis, John, "A letter of Dr. Wallis, dated Oxford Sept. 6. 1697. Containing some additions to his letter about thunder and a correction to the 109th Cap. of his Algebra," *Philosophical Transactions of the Royal Society of London*, 19 (1695–1697), no. 233, 729–730.

[51] Wallis, John, *The Arithmetic of Infinitesimals*, translation of Wallis 1656 by Jacqueline A. Stedall, Springer 2004