Ultra Square Palindromes

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All numbers below are non-negative, decimal integers n, with length (number of digits) $\ell = \ell(n)$, and decimal representation $n = n_\ell ... n_1$ ($n_\ell \neq 0$). Write Σn for the sum of the digits of n, Πn for the product of its digits.

Recall that n is a (numerical) *palindrome* if it is invariant under reversal of digits: $n = n_\ell ... n_1 = n_1 ... n_\ell$. To avoid trivialities, we require $\ell(n) > 1$. A palindrome s is *square* if $s = r^2$ for some integer *root* $r > 1$. The online site [1] provides a wealth of information on square palindromes (SPs), including a complete list of all instances from 3 to 32 digits. Keith [2] introduced a partition of the collection of all SPs into four infinite families, **E**, **B**, **T**, and **A**, with prescribed designs, and one additional *sporadic* class **S**, which may or may not be infinite, of seemingly irregular examples. It turns out that all but the sporadic SPs must have odd length. Some familiarity with these two sources is assumed here.

Our goal is to investigate SPs with two additional "square properties," and to show there are only 18 non-sporadic examples of any size with these properties, only 3 known sporadic examples with less than 55 digits, and no known additional ones. We will also explain why any more examples beyond these 21 are likely to be extremely rare. So, let us call a non-trivial palindrome s *ultra square* if

- s is a square,
- Σs is a square, and
- Πs is a square > 0.

Note that if some digit $s_k = 0$, then Πs is the trivial square 0. In pursuit of a more rarified collection, we limit ultra square palindromes (*USP*s) to those with *all digits positive.*

To begin our investigation of USPs we prove a result that is fundamental to most of the arguments that follow, and which is assumed in [1] and [2].

Proposition. Let r be a palindrome, and assume there are *no carries* in the computation of its square $s = r^2$. (We call such an r *carry-free.*) Then the following properties hold for r and s:

- $\ell(s) = 2\ell(r) - 1$,
- s is a palindrome,
- $s_{\ell(r)} = \Sigma r_k^2$, and
- $\Sigma s = (\Sigma r)^2$.

In particular, if r consists entirely of 1s and 0s, then $s_{\ell(r)} = \Sigma r$ = the number of 1s in r.

Proof. Let ℓ denote the length of r. Since there are no carries in the computation of $s = r^2$, it has $2\ell - 1$ digits, and the following are immediate consequences of that computation: for $1 \leq k \leq \ell$,

$$s_k = r_1 r_k + r_2 r_{k-1} + \ldots r_k r_1, \quad s_{2\ell-k} = r_\ell r_{\ell-k+1} + r_{\ell-1} r_{\ell-k+2} + \ldots + r_{\ell-k+1} r_\ell.$$

Here the first sum represents the total of all contributions to the kth decimal place of s for positions k from rightmost to center, while the second sum computes values at positions from leftmost to center. By the palindrome property for r: $r_k = r_{\ell-k+1}$ $(1 \leq k \leq \ell)$, corresponding terms in the two sums are equal, and hence s is a palindrome. Of course the two sums are mirror images for the central term s_ℓ, and

$$s_\ell = r_1 r_\ell + \ldots r_\ell r_1 = r_1^2 + \ldots + r_\ell^2.$$

If r has only 1s and 0s, then $r_k^2 = r_k$ for all k, hence $s_\ell = \Sigma r$. The carry-free assumption asserts that none of the s_k exceed 9, so each digit is determined independently.

A geometric representation of the convolutions for s_k and $s_{\ell-k}$ in the previous paragraph is particularly helpful to establish the final bullet of our Proposition. Start with a discrete Cartesian grid of the ℓ^2 points $[1, \ell] \times [1, \ell]$. Imagine labeling point (i, j) in this "times table" with the value $r_i r_j$. Then the first convolution above is obtained by summing labels along the diagonal $\{(i, j) : i + j = k\}$, the second along the diagonal $\{(i, j) : i + j = 2\ell + 1 - k\}$. Since Σs is the sum over all labels on any diagonal of the grid, i.e., the entire grid, it follows that

$$\Sigma s = (\Sigma r)^2.$$

□

We will now explore Keith's five classes of SPs in search of USPs. All members of three of the four non-sporadic infinite families, **E**, **B**, and **T**, are generated by carry-free root palindromes, so our Proposition applies and Σs is square. Members of the fourth infinite family **A** are generated by more mysterious, asymmetric roots with central digit 9, but again it turns out that Σs is always square. Thus, for any non-sporadic SP the sum condition holds. As already mentioned, all non-sporadic s have odd length $2\ell - 1$, where $\ell = \ell(r)$, so we can write

$$\Pi s = (s_1 \cdot \ldots \cdot s_{\ell-1})^2 \cdot s_\ell.$$

Therefore, the product will be square if and only if the central digit s_ℓ is square. But $s_\ell \geq r_1^2 + r_{\ell-1}^2 \geq 2$ when our Proposition applies. Thus, *a non-sporadic SP is a USP if and only if it has all positive digits and central digit* 4 *or* 9.

As a warm-up in our USP search, let's consider Keith's family **E** of *Even* root SPs, such that r has the form 2[0]2 where [0] is a block of 0 or more 0s of even length, or $2[0]x[0]2$, where [0] is any block of 0 or more 0s and x is 0 or 1. These roots are all carry-free, and so generate SPs, but there are only two positive cases: 22, and 212. $22^2 = 484$, with central digit 8, so this is not a USP. But $212^2 = 44944$ has central digit 9, so this is our first find:

# 1	**[12]**	44944	212	25	2304

The format here is: USP entry index, SP entry index from [1], $s, r, \Sigma s, \Pi s$. In fact, this is the smallest of all USPs, and the 12th smallest SP (the four trivial one-digit squares included).

Having examined root palindromes for SPs with outermost digits 2, we note that 1 is the only other carry-free possibility. Of course 0 is not allowed as lead digit, whereas $r_\ell = r_1 \geq 3$ forces $s_\ell \geq 18$. Moreover, r must take the form $1[x']y[x]1$, where x' is the reverse of x, for some some string x of length 0 or more consisting of only 1s and 0s, and central digit y is $0, 1$, or 2. Otherwise, it is easy to check that the s_ℓ computation produces a carry. In particular, the only possible r containing a 2 are of the form $1[x']2[x]1$, where said string x is either empty or consists of a single 1, or once more, $\Sigma r_k^2 > 9$. Thus, the only two candidates for USP roots generate squares with central digit 6 and 8, and neither is a USP. To sum up, the only USP with a carry-free palindrome root containing 2 as a digit is #1 above.

The remaining USPs generated by palindrome roots r must take the form $1[x]1$, where x is a string of length 0 or more with all digits 0 or 1. These roots comprise Keith's *Binary* (**B**) root family and the portion of his *Ternary* (**T**) root family not addressed in the previous paragraph. We will now develop a systematic method to find all 16 USPs of this form. Recall that the central digit of s must be 4 or 9, and that by the last property of our Proposition, this value agrees with the total number of 1s in r. Also observe that for s to have all positive digits, the second and second-to-last digits of r must also be 1; otherwise, e.g., $s_2 = r_1 \cdot r_2 + r_2 \cdot r_1 = 0$.

Consider first the case $\Sigma r = 4$. The smallest possible root is $r = 1111$, with $s = 1234321$. Next comes $r = 11011, s = 121242121$. These are both USPs, so we add them to our collection:

# 2	**[17]**	1234321	1111	16	144
# 3	**[24]**	121242121	11011	16	64

However 110011^2 has a 0 in places 4 and 8, as will the square of a root with any longer central string of 0s. So these two are the *only* non-sporadic USPs with central digit 4.

Turning now to the case $\Sigma r = 9$, we describe an algorithm to determine all 14 USPs with roots r of the form $11(x')1(x)11$, where each digit of string x is 0 or 1, exactly two digits are 1s, and $s = r^2$ has all digits positive. The idea is to develop the right half of r, right to left from digit 1 to digit $\ell - 1$, in terms of a pruned binary tree. Begin with $1 \leftarrow 1$, and then include directed edges $1 \leftarrow$ until a total of four 1s is reached, and $0 \leftarrow$ as long as the square of the evolving root $r_1 \ldots r_k 1 r_k \ldots r_1$ has all digits positive. Fig.1 gives a representation of the resulting tree, rotated 90° counterclockwise.

The smallest USP generated by a root with all digits 1 or 0 and a total of nine 1s has right end 1111, corresponding to the leftmost branch of the graph stopped after four 1s. There are three consecutive 0s afterward, indicating that from one to three 0s can be inserted before the central 1 to generate three more *USP*s. The leftmost branch terminates after three 0s because adjoining a fourth would create a 0 in the eighth digit of s. As one more illustration of Figure 1, consider the longest branch in the tree, 11100100, corresponding to $r = 11100100100100111$. This give rise to a 33-digit USP included only in the *Plain Text Squares* table of [1], which extends the exhaustive list from 32 up to 43 digits. Adding one more 0 on both sides of the central 1 introduces a 0 in r^2, so this is the largest USP captured by the tree. For currently the most extensive complete listing of SPs, up to 45 digits, see [3].

From the levels of our tree it is easy to read off the lengths of all USPs identified: one of length 17, three of length 21, 5 of length 25, four of length 29, and one of length 33, for a total of 14. The results are tabulated as follows:

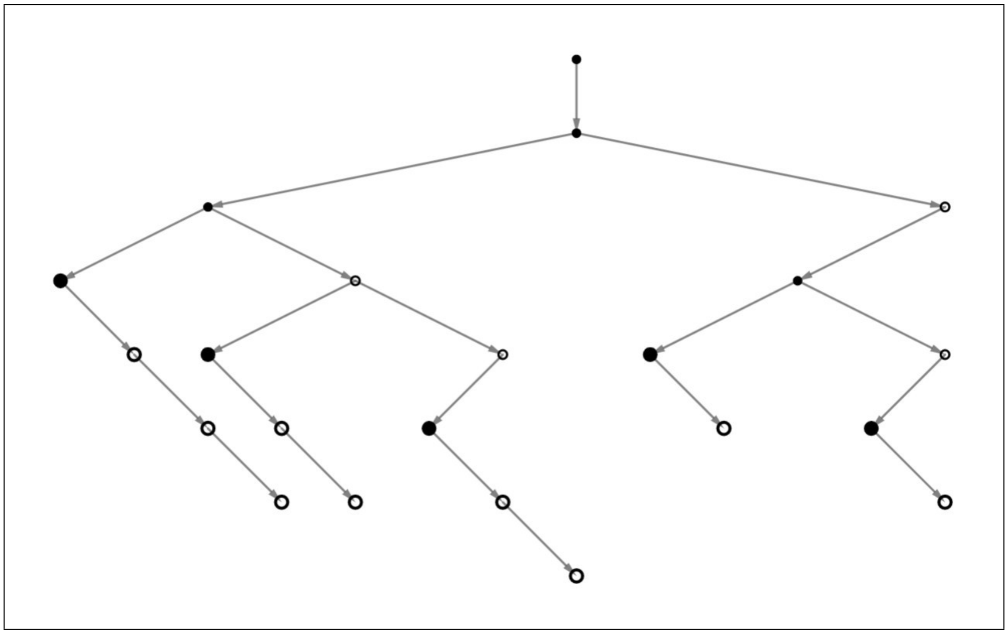

Figure 1: The right edge for 14 USPs with roots r of the form $11(x')1(x)11$, and $\Sigma r = 9$. Bullets with black and white centers denote 1s and 0s, resp. Larger bullets show possible leftmost digits of x.

#	5	**[92]**	12345678987654321	111111111	81	14631321600
#	7	**[148]**	121244565696565442121	11011111011	81	29859840000
#	8	**[153]**	123234645696546432321	11101110111	81	96745881600
#	9	**[155]**	123434346696643434321	11110101111	81	139314069504
#	11	**[246]**	1212225444549454445222121	1101011101011	81	94371840000
#	12	**[248]**	1212423436449446343242121	1101101011011	81	110075314176
#	13	**[253]**	1232124642369632464212321	1110011100111	81	61917364224
#	14	**[255]**	1232324252649462524232321	1110101010111	81	171992678400
#	15	**[257]**	1234323224469644223234321	1111001001111	81	247669456896
#	16	**[397]**	12122232425262926252423222121	110101010101011	81	76441190400
#	17	**[406]**	12321224243244944234242212321	111001010100111	81	195689447424
#	18	**[408]**	12323222322444944422322232321	111010010010111	81	440301256704
#	19	**[410]**	12343212222246964222221234321	111100010001111	81	110075314176
#	20	**[628]**	12321222223224249424232222212321	11100100100100111	81	195689447424

The final and most exotic infinite family, **A**, in Keith's classification consists of *asymmetric* root SPs. The roots for these obey no-carry to the right of center, but have one or more consecutive 9s from the central digit leftward. The general format is

$$1(x')0[9]9[0]1(x)1,$$

where [9] is a string of 9s of length 0 or more, [0] a corresponding string of 0s of the same length, and (x) is a string of 0s or 1s of length at least 1, (x') its reverse. In contrast to the other three infinite families, however, not every string with this format is an SP. Additional and quite complex requirements were identified in [2] and by David W. Wilson through correspondence described in reference [1]. For our purposes, we only need to know that (x) contains at most two 1s, in order to preserve no carry right of center in root r.

The two smallest asymmetric SPs are 1230127210321, with root 1109111, and 12120030703002121, with root 110091011. Of course, neither is a USP since both contain a 0. Note, however, that in both cases $\Sigma s = (\Sigma_1 r)^2 = 25$, where Σ_1 denotes the number of digits equal to 1. The next smallest asymmetric SP turns out to be the *last* in our accounting for all 18 non-sporadic USPs:

# 4	**[90]**	12341234943214321	111091111	49	2985984

Again $\Sigma s = (\Sigma_1 r)^2$, and this seems to be true for any asymmetric SP, but we will now show that #4 is the only member of family **A** with all positive digits. The key is the following

Claim. Let z be a string of 1s and 0s of length at least 10, with first (rightmost) digit 1, and at most three additional 1s. Then the first ten digits of z^2 contain at least one 0.

Proof. The first k digits of z^2 are determined by the first k digits of z, so we do a systematic check. As already noted, a right edge of 01 in z produces the same in z^2. Thus we need only check z beginning with 11. Now if z begins with 0011, then there must be a 0 as the fourth digit of z^2. We handle the strings z starting with 111 and those starting with 1011 separately. In the former case, $z =$ _______111, if the last possible additional 1 is in one of the four leftmost digits (7-10) or not present, then z^2 has a 0 as the sixth digit. Or if the fourth 1 is in place $4, 5$, or 6, then 0 appears in place $8, 8$, or 9 of z^2, respectively. Otherwise, for $z=$ ______1011, if the fourth 1 is in one of the two leftmost digits (9-10) or not present, then z^2 has a 0 as the eighth digit. Or if the fourth one is in place $5, 6, 7$, or 8, then 0 appears in place $10, 8, 6$, or 6, respectively. $\square$

Suppose now that r is the root for an asymmetric SP s of length at least 21. Then the first ten digits satisfy the hypothesis of the Claim, so s contains a 0 digit and cannot be a USP. A dedicated web page at [2] provides a separate listing of all asymmetric SPs with roots up to length 23. There are 43 with roots having lengths up to 19, and careful inspection confirms that #4 above is the only one with all digits positive, and so the unique USP in **A**.

It remains only to discuss the matter of sporadic USPs. Recall that these are formally defined as those that do not belong to any of Keith's four infinite families. They arise from seemingly arbitrary roots r that, when squared, cause carries in various decimal places and somehow produce mirror symmetry as the result. For instance, the smallest odd- and even- length sporadic SPs are $26^2 = 676$ and $836^2 = 698896$, respectively. For a list up to 55 digits, separated into odd and even cases, see [1]. Curiously, there are roughly four times as many odd examples as even. We will now identify all the sporadic USPs, splitting the search according to length parity.

For SPs of odd length to be USPs, they must have all positive digits and central digit $1, 4$, or 9. A quick check of the 76 known odd sporadic SPs with at most 55 digits yields strings of length $7, 13, 15, 19, 21, 23$, and 27 with these properties. For only the two of lengths 19 and 21 do the digit sums turn out to be squares. The latter is our only USP with a central 1.

For SPs of even length with all positive digits, their product is automatically square. However, there are only 18 known even sporadic SPs up to 55 digits. Only four of these, with lengths $6, 12$, and 36 (twice), have all digits positive, and only one of the two length-36 strings has digits that sum to a square, making it the largest USP in our collection and the only even one.

In summary, we can report that there are only three known sporadic USPs with at most 55 digits:

6 [**113**] 6158453974793548516 2481623254 100 3292047360000

10 [**156**] 184398883818388893481 13579355059 121 112717121716224

21 [**741**] 632914544142271449944172241445419236 795559265009384106 144 2796089100573081600

These bring the size of our complete USP collection to 21.

Are there more? From 3 to 55 digits, the number of known sporadic SPs fluctuates between 1 and 5, except for several even digits with none, and a record of 6 for length 21. This is neither compelling evidence for a largest SP, nor for ever more examples with longer lengths. It *does* seem reasonable to conjecture a longest length for any square palindrome with all digits positive since the disordered nature of sporadic roots, subject to complex effects of carried digits, should make it exceedingly unlikely for a very long square to have no 0s. The additional requirements for USPs then make them even more unlikely. While we would not be inclined to wager that there are only 21 USPs, it does seem most probable that if there are any more at all, they are few and far between. Perhaps this account, and the prospect of higher speed computation in the future, will motivate further exploration of very long square palindromes.

References

[1] P. De Geest, Palindromic Squares, *World of Numbers* (http://worldofnumbers.com/square.htm).

[2] M. Keith, Classification and Enumeration of Palindromic Squares, *J. Recreational Math.* **22:2** (1990) 124-132. (An online copy is available here: https://oeis.org/A002778/a002778_1.pdf.)

[3] OEIS Foundation Inc., Entry A002778 in The On-Line Encyclopedia of Integer Sequences (2022), (https://oeis.org/A002779/b002779.txt).