4 Acute and Isosceles Triangles

An acute triangle is one with each interior angle less than 90 degrees. What is the smallest number of nonoverlapping acute triangles into which a square can be divided?

I asked myself that question some twenty years ago, and I solved it by showing how to cut a square into eight acute triangles as is indicated in Figure 98, top. Reporting this in a column, reprinted as Chapter 3 of my *New Mathematical Diversions from Scientific American* (Simon and Schuster, 1966), I said: "For days I was convinced that nine was the answer; then suddenly I saw how to reduce it to eight."

Since then I have received many letters from readers who were unable to find a solution with nine acute triangles but who pointed out that solutions are possible for ten or any higher number. The middle illustration in Figure 98 shows how it is done with ten. Note that obtuse triangle *ABC* is cut into seven acute triangles by a pentagon of five acute triangles. If *ABC* is now divided into an acute and an obtuse triangle by *BD*, as is indicated in Figure 98, bottom, we can use the same pentagonal method for cutting the obtuse triangle *BCD* into seven acute triangles, thereby producing eleven acute triangles for the entire square. A repetition of the procedure will produce 12, 13, 14, . . . acute triangles.

Apparently the hardest dissection to find is the one with nine acute triangles. Nevertheless, it can be done.

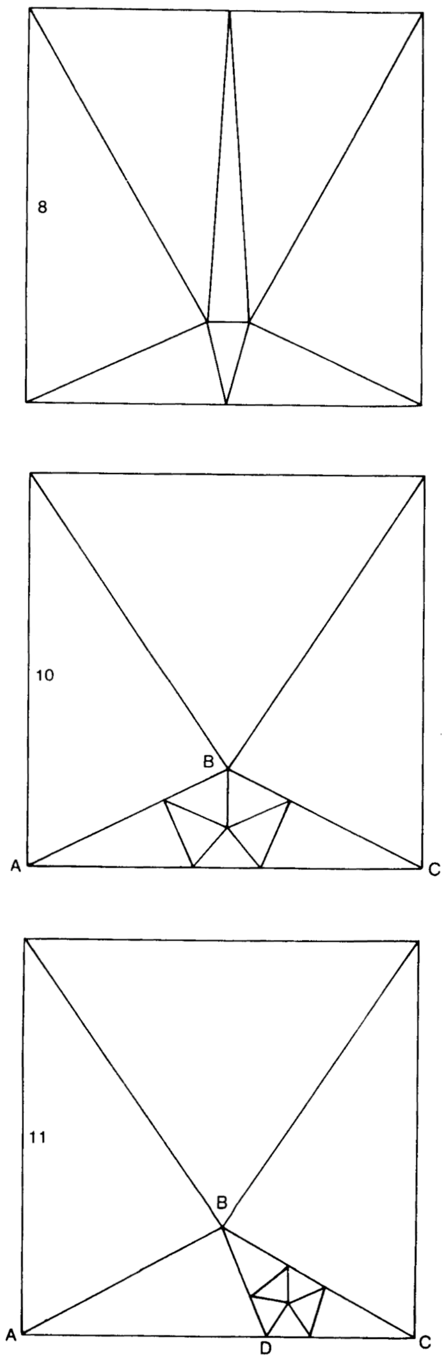

Figure 98

There are many comparable problems about cutting figures into nonoverlapping triangles, of which I shall mention only two. It is easy to divide a square into any even number of triangles of equal area, but can a square be cut into an odd number of such triangles? The surprising answer is no. As far as I know, this was first proved by Paul Monsky in *American Mathematical Monthly* (Vol. 77, No. 2, pages 161–164; February 1970).

Another curious theorem is that any triangle can be cut into n isosceles triangles provided n is greater than 3. A proof by Gali Salvatore appeared in *Crux Mathematicorum* (Vol. 3, No. 5, pages 134–135; May 1977). Another proof, by N. J. Lord, is in *The Mathematical Gazette* (Vol. 66, pages 136–137; June 1982).

The case of the equilateral triangle is of particular interest. It is easy to cut it into four isosceles triangles (all equilateral) or into three isosceles triangles. (Some triangles cannot be cut into three or two isosceles triangles, which is why the theorem requires that n be 4 or more.) Can you cut an equilateral triangle into five isosceles triangles? I shall show how it can be done with none of the five triangles equilateral, with just one equilateral, and with just two equilateral. It is not possible for more than two of the isosceles triangles to be equilateral.

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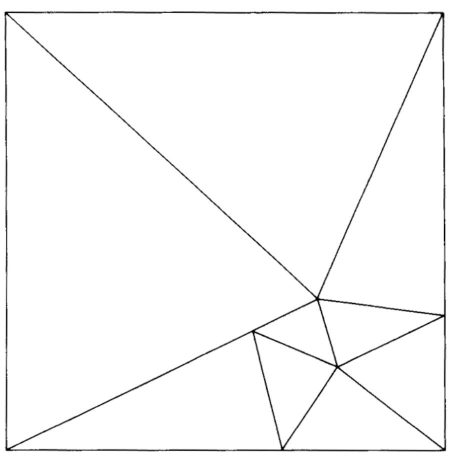
Figure 102 shows how to cut a square into nine acute triangles. The solution is unique. If triangulation is taken in the topological sense, so that a vertex is not allowed to be on the side of a triangle, then there is no solution with nine triangles, although there is a solution for eight triangles, for 10 and for all higher numbers. This curious result has been proved in an unpublished paper by Charles Cassidy and Graham Lord of Laval University in Quebec.

Figure 103 shows four ways to cut an equilateral triangle into five

Figure 102



isosceles triangles. The first pattern has no equilateral triangle among the five, the second and third patterns both have one equilateral triangle and the fourth pattern has two equilateral triangles. The four patterns, devised by Robert S. Johnson, appear in *Crux Mathematicorum* (Vol. 4, No. 2, page 53; February 1978). A proof by Harry L. Nelson that there cannot be more than two equilateral triangles is in the same volume of the journal (No. 4, pages 102–104; April 1978).

The first three patterns of Figure 103 are not unique. Many readers sent alternate solutions. The largest number, 13, came from Roberto Teodoro Garrido, a civil engineer in Buenos Aires.

Figure 103